Birzeit University
Math. Dept.
Math. 333

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Final Exam Solutions **First Semester 2015/2016**

Student Name:________________________________**Number:**___________ **Section**____

Q1:40 points Mark each of the following by True or False

——T- 1) If E is a nonempty subset of Z that is bounded from below, then $infE \in E$.

—–F– 2) A sequence $\{x_n\}$ converges to a if and only if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to a.

—-T- 3) Every nonempty bounded subset of Z has a maximum and a minimum.

——F– 4) If $\{x_n\}$ and $\{y_n\}$ converge to a and b, respectively, and $x_n < y_n$ for every $n > N$, for some fixed N, then $a < b$.

—F— 5) Every sequence of real number has a convergent subsequence.

—–T- 6) If A and B are nonempty subsets of R, $A \subseteq B$, and B has an infimum, then $inf(A) \geq inf(B)$.

—–T– 7) If A and B are nonempty subsets of R, $A \subseteq B$, and B has a supremum, then $sup(A) \leq sup(B)$.

—–T— 8) If $\{x_n\}$ is a convergent sequence. Then $\{x_n/n\}$ converges.

—-T—- 9) If (x_n) converges to 0, and $x_n > 0$ then $(1/x_n)$ diverges to infinity.

——F– 10) If (x_n) has a convergent subsequence, then (x_n) is bounded.

——T- 11) $|a - b| \geq |a| - |b|$

—–F— 12) If $x < b$ then $x^2 < b^2$

—–F— 13) The solution of $\frac{2x}{x+1} \leq 1$ is $x \geq 1$.

—–F— 14) If (x_n) is a monotonic sequence, then (x_n) converges.

—-T— 15) If (x_n) is a cauchy sequence, then (x_n) is bounded.

—–T— 16) (x_n) converges to a iff $(x_n - a)$ converges to 0.

——T– 17) If (x_n) is bounded, then $(\frac{x_n}{n})$ converges to 0.

—-T—- 18) $\sqrt{3n+2} - \sqrt{n}$ diverges.

—–T— 19) $(\frac{lnn}{n})$ converges to 0.

——F– 20) If (x_n) is bounded, and (y_n) converges then $(x_n y_n)$ converges.

——F– 21) If (x_n), and (y_n) are cauchy then $(\frac{x_n}{y_n})$ is cauchy.

——F– 22) If $a, b, c \in R, a < b$, then $|a + c| < |b + c|$.

—–T— 23) If $a > 2, b = 1 + \sqrt{a - 1}$, then $b < a$.

—-F— 24) If $A = \{1 - \frac{(-1)^n}{n} : n \in N\}$ then $supA = 1, infA = 0$

——T– 25) A function $f : E \subseteq R \to R$ is continuous at a point $a \in E$ iff $lim_{x \to a} f = f(a)$.

——F– 26) A function $f : E \subseteq R \to R$ is discontinuous at a point $a \in E$ iff there exists a sequence x_n in E such that $f(x_n)$ does not converge to $f(a)$.

——-F- 27) If f, g are uniformly continuous then fg id uniformly continuous

——F– 28) If E is a bounded interval and f is continuous on E, then f is bounded.

—–T– 29) If E is a bounded interval and f is uniformly continuous on E, then f is bounded.

——F– 30) If $f = g^2$, and f is differentiable then g is differentiable.

—–T— 31) If E is a bounded interval and f is continuous on E, then f is integrable on E.

—–T- 32) If f, g are increasing functions then fog is increasing.

——F– 33) If f, g are differentiable functions such that $f' = g'$ then $f = g$.

——F– 34) If $|f|$ is a differentiable function then f is differentiable.

——F– 35) If $|f|$ is integrable on E then f is integrable on E.

—–T– 36) If f is differentiable and f is even then f' is odd.

——T– 37) If f is differentiable on $[a, b]$ then f is uniformly continuous.

——T– 38) If f is differentiable and f is odd then f' is even.

——F– 39) If f is integrable on $[0, 1]$, then f is continuous

—–F— 40)If $a \in R, A \subseteq R$, then sup$aA = asup(A)$.

—–F– 41) If f is differentiable on (a, b) then f is uniformly continuous on (a, b).

—–F– 42) If $A \subseteq R$ is bounded, then $i = inf(A)$ iff for any $\epsilon > 0$ there exists $x \in A$ such that $x - \epsilon < i$.

Q2 (70 points). This part consists of 12 Questions, answer any 10 and only 10 of them. Explain every step and write clearly

(1) If x_n is a sequence of real numbers, $x \in R$ and $b_n \in R$ converges to 0 such that $|x_n - x| < b_n$ for all $n \in Z^+$. Prove that x_n converges to x.

Let $\epsilon > 0$ be given, since b_n converges to 0, so there exists $N \in Z^+$ such that $|b_n - 0| < \epsilon$, for all $n \geq N$, and since $|x_n - x| < b_n$ for all $n \in Z^+$. So $|x_n - x| < \epsilon$ for all n $geqN$, so x_n converge to x.

(2) If f is continuous on R and $f(q) = 0, \forall q \in Q$. Show that $f(x) = 0$ for all $x \in R$.

By density of rational numbers for any $x \in R$, there exists a sequence of rational numbers q_n converges to xm, and by continuity (sequential continuity), $f(x) = limf(q_n) = f(limq_n) = 0$

(3) If f, g continuous on $[a, b]$ and $f(a) < g(a), f(b) > g(b)$. Show that there exists $c \in (a, b)$ such that $f(c) = g(c)$.

Apply Roll's Theorem to $h = f - g$

(4) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$ but it is uniformly continuous on $[1, 2]$.

$x_n = \frac{1}{n}$ is cauchy in $(0, 1)$ but $f(x_n) = n$ is not cauchy, so f is not uniformly continuous on $(0, 1)$

$[1, 2]$ is closed and bounded and f is continuous on $[0, 1]$, so f is uniformly continuous on $[0, 1]$

(5) If $f : R^+ \to R^+$ is a function such that $f(\frac{x}{y}) = f(x) - f(y)$ for all $x, y \in R^+$. Prove the following

(a) f is continuous if f is continuous at 1.

First notice that $f(1) = 0$. Let $x > 0$, and y_n a sequence of positive real numbers that converges to x, then $\frac{y_n}{x}$ converges to 1 and by continuity of f at 1, and $f(y_n) - f(x) = f(\frac{y_n}{x})$ converges to $f(1) = 0$, so $f(y_n)$ converges to $f(x)$

(b) f is differentiable if f is differentiable at 1.

Similar to (a), assume f is differentiable at 1, $f'(x) = \frac{f(x+h)-f(x)}{h} = \frac{f(\frac{x+h}{x}}{h} = \frac{1}{x}\frac{f(1+\frac{h}{x})}{h/x} \to \frac{f'(1)}{x}$

(c) If f is differentiable, and $f(1) = 0$. What can you say about f.

$f'(x) = a/x$, where $a = f'(1)$

(6) If $f : [a, b] \to R$ is continuous, $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$ for all $x, y \in [a, b]$. Show that there exists $c \in [a, b]$ such that $f(c) = c$.

Let $x_0 \in [a, b]$, and define a sequence x_n by $x_1 = f(x_0), x_{n+1} = f(x_n)$, then show x_n is cauchy, so x_n converges, say, to x^*. Show x^* is a fixed point, that is $f(x^*) = x^*$, why?. Moreover, you can show it is unique, why?.

(7) If f is differentiable on R and $f(0) = 1, |f'(x)| < 1, \forall x \in R$.

Show that $|f(x)| \leq |x| + 1, \forall x \in R$.

Apply MVT to f on $[0, x]$

(8) Show that $x + 1 \leq e^x, \forall x \in R$.

for $x > 0$, consider $f(x) = e^x - x$, this is increasing on $[0, \infty)$, so $f(x) = e^x - x > f(0) = 1$, so $e^x - x > 1$ for $x \geq 0$.

for $x < 0$, consider $f(x) = e^x - x$, this is decreasing on $(-\infty, 0]$, so $f(x) = e^x - x > f(0) = 1$, so $e^x - x > 1$ for $x \leq 0$.

(9) If f, g differentiable on $[a, b]$ and $f'(x) < 1 < g'(x), \forall x \in [a, b]$.

Show that $|f(x) - f(a)| \leq |g(x) - g(a)|, \forall x \in [a, b]$.

Apply GMVT

(10) If $f : [0, 2] \to R$ is defined by $f(x) = 1$ if $0 \leq x \leq 1$, $f(x) = 2$ if $1 < x \leq 2$. Show that f is integrable and find $\int_0^2 f(x)dx$.

See notes, take a partition $P = \{0, x_1, 1, x_2, 2\}$, then take $U(f, P)_L(f, P) < \epsilon$

or f is continuous on $[0, 2]$ except at 1, and bounded on $[0, 2]$, so f is integrable

(11) If $f : [0, 2] \to R$ is continuous. Show that f is integrable.

See notes or book

(12) If $f : [0, 2] \to R$ is a nonnegative continuous function, and $\int_0^2 f(x)dx = 0$. Show that $f = 0$.

by contradiction, assume not say there exists $c \in [0.2]$ such that $f(c) > 0$, by continuity there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$. So

$\int_0^2 f(x)dx > \int_{c-\delta}^{c+\delta} f(x)dx > 0$.

(13) **(a)** If $f : [0, 2] \to R$ is a bounded function, and RI. Show $U \int_0^2 f(x)dx = L \int_0^2 f(x)dx$.

See notes

(b) If $f : [0, 2] \to R$ is a bounded function. Write the definition of RI.

See notes

(c) If $f : [0, 2] \to R$ is a bounded function, and RI. Write the definition of $U \int_0^2 f(x)dx$ and $L \int_0^2 f(x)dx$.

See notes